Dynamical Systems

Exam 4 April 2022

university of groningen

The exam consists of ~~4~~ 5 questions. You have 120 minutes to do the exam. Students eligible for extra time get an additional 20 minutes at the end of the exam. You can achieve 100 points in total which includes a bonus of 10 points.

1. **[4+6=10 Points]**

   Each of the following time-continuous systems depend on a parameter $a \in \mathbb{R}$. Sketch the bifurcation diagram including representative phase portraits and classify the bifurcations of equilibrium points.

   (a) The one-dimensional systems $x' = x \cos x + ax$.

   (b) The planar systems

   $$\begin{aligned} r' &= r - r^3, \\ \theta' &= a + \cos\theta, \end{aligned}$$

   where r and θ are polar coordinates. In this case show the bifurcation diagram in the form θ versus a.

2. **[20 Points]**

   Consider the planar systems

   $$X' = \begin{pmatrix} b & a \\ -1 & b \end{pmatrix} X$$

   with parameters $a, b \in \mathbb{R}$. Sketch the regions in the $a-b$ plane where this system has different types of canonical forms. In each region give the canonical form and sketch the phase portrait of the system in canonical form.

3. **[3+4+8=15 Points]**

   Consider the planar system

   $$\begin{aligned} x' &= x - y - x(x^2 + y^2), \\ y' &= x + y - y(x^2 + y^2). \end{aligned}$$

   (a) Show that $(x, y) = (0, 0)$ is the only equilibrium point.

   (b) Let $r(x, y) := x^2 + y^2$ and $\dot{r}(x, y) := r_x(x, y)x' + r_y(x, y)y'$. Show that for $(x, y) \in \mathbb{R}^2$ with $r(x, y) > 1$ it holds that $\dot{r}(x, y) < 0$, and similary for $(x, y) \in \mathbb{R}^2$ with $r(x, y) < 1$ it holds that $\dot{r}(x, y) > 0$.

   (c) State the Poincaré-Bendixson Theorem and use the results in parts (a) and (b) to argue that the system must have a limit cycle.

– please turn over –

4. **[2+6+4+13=25 Points]**
Consider the pendulum

$$x' = y,$$
$$y' = -\nu y - \sin x,$$

where $\nu \geq 0$ is a friction parameter and $(x, y) \in S^1 \times \mathbb{R}$ where we view the circle S^1 to be the interval $[-\pi, \pi]$ with the boundaries identified.

(a) Show that the system has two equilibrium points located at $(x_0, y_0) := (0, 0)$ and $(x_1, y_1) := (\pm\pi, 0)$.

(b) Show from the linearization that the equilibrium point at (x_1, y_1) is a saddle and for $\nu > 0$, the equilibrium point at $(x_0, y_0) = (0, 0)$ is a sink.

(c) Show that for $\nu = 0$, the system is Hamiltonian by constructing a Hamilton function H.

(d) Show that for $\nu \geq 0$ and $0 < h < 2$, $H(x, y) - H(0, 0)$ is a Lyapunov function for the equilibrium at (x_0, y_0) in the region $D_h = \{(x, y) \in S^1 \times \mathbb{R} \,|\, H(x, y) \leq h, |x| \leq \pi\}$ where H is the Hamilton function from part (c). State the Lasalle Invariance Principle and use it to show that for $\nu > 0$, the equilibrium at $(x_0, y_0) = (0, 0)$ is asymptotically stable with D_h belonging to the basin of attraction.

5. **[12+8=20 Points]**

(a) Show by direct proof (i.e. without using a conjugacy) that the tent map

$$t : [0, 1] \to [0, 1], \quad x \mapsto \begin{cases} 2x & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

satisfies all three conditions of Devaney's definition of chaos.

(b) Show that if two maps $f, g : [0, 1] \to [0, 1]$ are conjugate, and the discrete-time systems $x_{n+1} = f(x_n)$, $n \in \mathbb{Z}_{\geq 0}$, has dense periodic points, then the discrete-time system $x_{n+1} = g(x_n)$, $n \in \mathbb{Z}_{\geq 0}$, has also dense periodic points.